Effects of Surface Tension over a Flow Past a Flat Plate

Gathia P.W.[1*] and Lonyangapuo, P.[1]
Department of Mathematics
Egerton University, P. O. Box 536 – 20115 Egerton, Kenya.
Email: patricia_wanjiru@yahoo.com
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Abstract

We consider a free surface flow past a flat plate. We consider relations between the results of Anderson and Vanden-Broeck (1996) and those of Osborne and Stump (2000), and present new solutions. There is need to know the number of parameters needed to fix solutions uniquely. We show here that there is a three parameter family of solutions when the fluid is of finite depth. These solutions are characterised by a train of waves in the downstream region and by a discontinuity in slope at the separation point. The family includes a two parameter sub-family for which the free surface leaves the plate tangentially. It is shown that this sub-family reduces to the linear solutions of Osborne and Stump (2000) when the depth of submergence of the plate is small. Also, the three parameter family reduces to the one parameter family of Anderson and Vanden-Broeck (1996) as the depth of the water tends to infinity. Finally, fully nonlinear solutions with large capillary waves are presented.

Key words: Free surface, flat plate, surface tension, capillary waves, wavelength, waveless

Introduction

Free surface flows past surface objects pose difficult mathematical problems due to strong singularities that occur at the intersections of rigid walls with free surfaces. An example is the free surface flow generated by a ship moving at a constant velocity at the surface of water. The singularity takes the extreme form of sprays or jets at the bow of the ship. Here, a highly simplified geometry for which such singularities do not occur or are not known analytically. In this case we assume that the surface is a flat plate and that the free surface separates from the end of the plate, Figure 1.

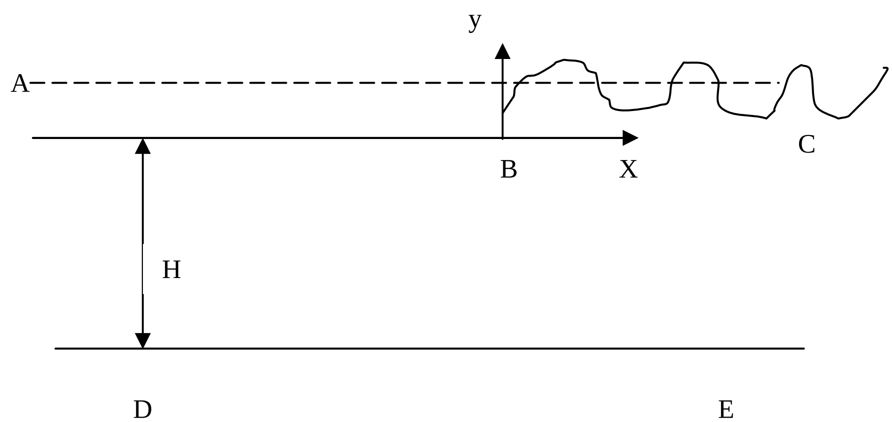

Figure 1: Diagrammatic representation of the flow

If the separation point was on top of the plate, then there would be a point inside the flow with an infinite velocity. Vanden Broeck (1984) included gravity and neglected surface tension. We use the conservation of momentum theorem to derive an exact relation between the steepness of the waves and the Froude number. Vanden-Broeck (1996) assumed that the depth is infinite and considered only the effect of surface tension, computed a one parameter family of solutions with capillary waves on the free surface. By using conservation of momentum, they showed that these solutions are characterised by a discontinuity in the slope of the free surface at the separation point. That is, the free surface does not leave the plate tangentially. Osborne and Stump (2000) considered the same problem in water of finite depth, linearized the equations and solved the resulting equations using the Weiner-Hopf techinique (1987). A two parameter family of solutions for which the free surface leaves the object tangentially was obtained.

In this paper, we consider relations between the results of Anderson and Vanden-Broeck (1996) and those of Osborne and Stump (2000), and present new solutions. We need also to know the number of parameters needed to fix solutions uniquely. It will be shown that there is a three parameter family of solutions when the fluid is of finite depth. These solutions are characterised by a train of waves in the downstream region and by a discontinuity in slope at the separation point. The family includes a two parameter sub- family for which the free surface leaves the object tangentially. It is shown that this sub-family reduces to the linear solutions of Forbes and Schwarz (1982) when the depth of submergence of the object is small. Also, the three parameter family reduces to the one parameter family of Anderson and

Vanden-Broeck (1996) as the depth of the water tends to infinity. Finally, fully nonlinear solutions with large capillary waves are presented.

Mathematical Formulation

We consider a semi-infinite horizontal surface *AB* in a uniform stream where the plate is at the fluid surface. The stream is finite in depth bounded by a horizontal bottom '*DE*, see Figure 1'. There is a free surface beginning where the plate *AB* ends and extending into the far field. The point *B* is the separation point and acts as the origin. The object *AB* is in the same horizontal plane as the *x*-axis. As $x \to -\infty$, the flow reduces to a uniform stream with velocity *U* and uniform depth *H*.

We neglect the effects of gravity and consider the effects of surface tension, so that

Bernoulli's equation gives:

$$\frac{1}{2}(u_x^2 + u_y^2) + \frac{p^*}{\rho} = C^{**} \qquad (1)$$

where u_x and u_y are the dimensional and vertical components of velocity respectively, ρ is the density, p^* is the fluid pressure and C^{**} is the dimensional Bernoulli's constant.

For this problem it is essential to find the shape of the free surface. For an element of free surface *ds*, if $ds \to 0$, then the said element can be considered as an arc of a circle with centre O and radius *R*, see Figure *2*.

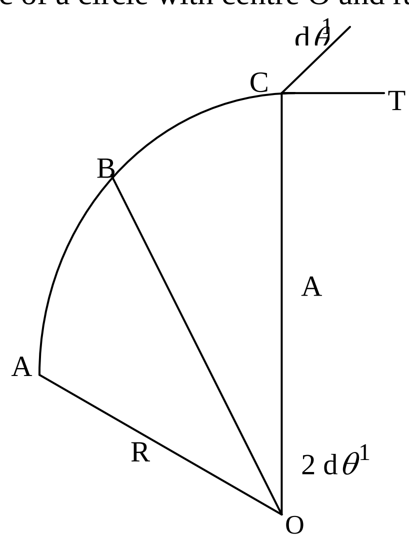

Figure 2: A section of a free surface

The surface element is bisected by *BO* such that $<AOB = <BOC = d\theta^1$. The resultant force due to surface tension τ acting on this element *ds* in the normal direction is the sum of the forces at the two end points, that is,

$$\text{Net force } = 2\tau \sin d\theta' \tag{2}$$

As $ds \to 0$, $d\theta' \to 0$ and so there is a force $2\tau \sin d\theta' \approx 2\tau d\theta'$ (3)

directed along *BO*. This force must be balanced by a force due to the pressure difference across the free surface. This is because the forces are acting on an element of zero mass. If we let *P* be the fluid pressure, P_a be the atmospheric pressure and equating force components along the normal *OB* gives:

$$2\tau d\theta' = (P - P_a)ds \tag{4}$$

since $ds \to 0$, then ds = 2R$d\theta'$ (5)

Substituting (5) into (4) gives:

$$(P - P_a)ds = \frac{\tau}{R}ds \text{ that is, } (P - P_a) = \frac{\tau}{R} = \tau k^* \tag{6}$$

where $k^* = \dfrac{1}{R}$ is the curvature.

Substituting (6) into (1), we obtain

$$\frac{1}{2}(u_x^2 + u_y^2) + \frac{\tau k^*}{\rho} = C^{**} \tag{7}$$

We now non-dimensionalize equation (7) by choosing *U* as typical velocity and *H* as a typical length so that equation (7) now becomes

$$\frac{1}{2}(u^2 + v^2) + \frac{k\tau}{\rho H U^2} = C^* \tag{8}$$

where $C^* = \dfrac{C^{**}}{U^2}$ is the dimensionless Bernoulli's constant.

The kinetic boundary conditions are

$$V = 0 \text{ on } AB \text{ and } DE \tag{9}$$

We also introduce a complex potential function, $f = \phi + i\psi$. Let $\phi = 0$ at *B* and $\psi = -1$ along the bottom *DE*. The image of the flow in the plane is an infinite strip (Figure 3).

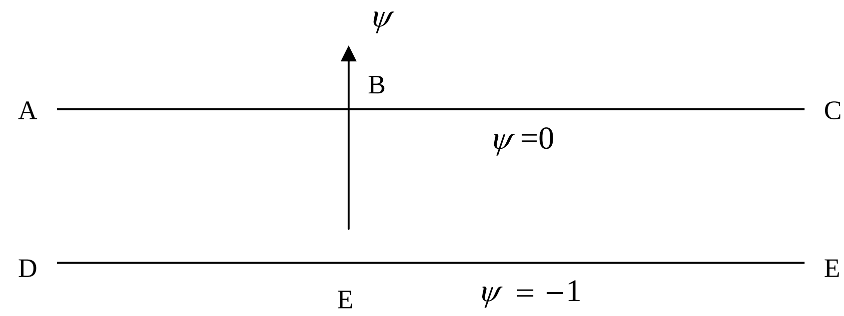

Figure 3: Flow representation in the f-plane

The kinematic boundary conditions (9) are now written as

$$v = 0 \text{ on } \psi = 0, \quad \phi < 0 \qquad (10)$$

and

$$v = 0 \text{ on } \psi = -1, \ -\infty < \phi < \infty \qquad (11)$$

We introduce a conformal mapping where

$$\eta = \alpha + i\beta = e^{-\pi f} = e^{-\pi(\phi + i\psi)} = e^{-\pi\phi}[\cos\pi\psi - i\sin\pi\psi] \qquad (12)$$

Below is Table 1 to show how the points in the Cartesian plane are transformed through the *f*-plane and η-planes.

Table 1: Comparison of *f*-plane and η-planes

Cartesian Plane	*f*-plane	η-planes
A	$\psi = 0, \phi = \phi_A = -\infty$	$\alpha = \alpha_A = \infty, \beta = 0$
B	$\psi = 0, \phi = \phi_B = 0$	$\alpha = \alpha_B = 0, \beta = 0$
C	$\psi = 0, \phi = \phi_C = \infty$	$\alpha = \alpha_C = 0, \beta = 0$
D	$\psi = -1, \phi = \phi_D = -\infty$	$\alpha = \alpha_D = -\infty, \beta = 0$
E	$\psi = -1, \phi = \phi_E = \infty$	$\alpha = \alpha_E = 0, \beta = 0$

It can be seen that the fluid boundaries *ABC and DE*, Figure 1, have been mapped onto the real axis of the η-plane, see Figure 4.

We introduce the complex velocity as u - iv to give:

$$\text{u - iv} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} = \frac{df}{dz} \qquad (13)$$

the complex velocity is also an analytic function of $z = x + iy$ since the derivative of an analytic function itself is analytic function. A new complex function, that is,

$\Gamma - i\theta = \ln(\frac{df}{dz})$ is introduced and is related to the complex velocity by

$$u - iv = e^{\Gamma - i\theta} \tag{14}$$

We use this to rewrite equation (8) in terms of the new variables Γ and θ giving

$$\frac{1}{2}e^{2\Gamma} - \delta e^{\Gamma}\frac{\partial\theta}{\partial\phi} = C^* \text{ on } BC \tag{15}$$

where $\delta = \dfrac{\tau}{\rho H U^2}$

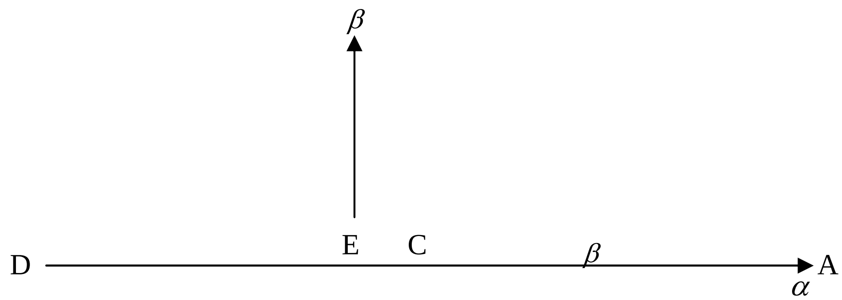

Figure 4: Flow region in the η plane on the upper

We now rewrite the kinematic boundary conditions (10) and (11) in terms of θ to fit in the η-plane as:

$$\theta = 0 \text{ for } \alpha > 1 \text{ and } \beta = 0 \tag{16}$$

and

$$\theta = 0 \text{ for } \alpha < 0 \text{ and } \beta = 0 \tag{17}$$

Applying Cauchy integral formula to the function τ -iθ, gives:

$$\Gamma(\alpha_0, 0) - i\theta(\alpha_0, 0) = \frac{1}{\pi i}\int_{\gamma} \frac{\Gamma(\alpha,\beta) - i\theta(\alpha,\beta)}{\eta - \alpha_0} d\eta \tag{18}$$

where α_0 is a point on the real axis, γ is a contour in the η plane consisting of the real axis, β=0, with a circular identation about the point α_0 and a half circle of arbitrary large radius extending into the upper half plane, Forbes *et al*., (1982). Letting the radius of the semi-circle to be R and taking limit as $R \to \infty$, equation (18) becomes,

$$\Gamma(\alpha_0,0)-i\theta(\alpha_0,0)=\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\Gamma(\alpha,0)-i\theta(\alpha,0)}{\alpha-\alpha_0}d\alpha \qquad (19)$$

where the integral is defined as a cauchy principle value. Considering only the real part of (19) and separating it into three integrals relating to the bottom *DE*, the plate *AB* and the free surface *BC* gives:

$$\Gamma(\alpha_0,0)=-\frac{1}{\pi i}\int_{-\infty}^{0}\frac{\theta(\alpha,0)}{\alpha-\alpha_0}d\alpha+\int_{0}^{1}\frac{\theta(\alpha,0)}{\alpha\text{-}\alpha_0}d\alpha+\int_{1}^{\infty}\frac{\theta(\alpha,0)}{\alpha-\alpha_0}d\alpha \qquad (20)$$

when applying the boundary conditions (16) and (17), it can be seen that no contribution is made by the first and last integrals, so equation (20) simplifies to

$$\Gamma(\alpha_0)=-\frac{1}{\pi}\int_{0}^{1}\frac{\theta(\alpha)}{\alpha\text{-}\alpha_0}d\alpha \qquad (21)$$

where

$$\Gamma(\alpha_0)=\Gamma(\alpha_0,0) \text{ and } \theta(\alpha_0)=\theta(\alpha\ \ ,0).$$

Considering the real part of equation (12) gives:

$$\alpha=e^{-\pi\phi}$$
$$d\alpha=-\pi e^{-\pi\phi}d\phi \qquad (22)$$

Using this change of variables in equation (21), then the differential equation to be solved is:

$$\frac{1}{2}e^{2\Gamma^1}-\tau e^{\Gamma^1}\frac{\partial\theta^1}{\partial\phi}=C^* \qquad (23)$$

with

$$\Gamma^1(\phi)=-\int_{0}^{\infty}\frac{\theta^1(\phi)e^{-\pi\phi}}{e^{-\pi\phi}-e^{-\pi\phi_0}}d\phi \qquad (24)$$

where

$$\Gamma^1(\phi)=\Gamma(e^{-\pi\phi}) \text{ and } \theta^1(\phi)=\theta(e^{-\pi\phi}).$$

Numerical Methodology

The aim is to calculate τ at points along the free surface in terms of θ by evaluating the integral in equation (23) numerically. These values are substituted in equation (23) to create a system of nonlinear equations for θ that are then solved by Newton iterations.

If we consider the parameters of the problem, the calculation of Γ used a uniform mesh, created in the *f*-plane. The mesh covered the region representing the free surface, (King *et al* 1987). The family of solutions with a discontinuity at the separation point *B*, that is, $\theta_1 \neq 0$ depend on the following three parameters: Fixing the surface tension parameter, δ, setting a value to the separation angle, θ_1, a third parameter is the mean water level, *d*, of the waves in the far field, figure 1. For an arbitrary wave, Figure 5, the mean water level can be evaluated by the formula, Hoffman (1992):

$$d = \frac{1}{\lambda}\int_{l_1}^{l_2} y dx \qquad (25)$$

Therefore, we need to know the wavelength of the free surface profile before doing any calculations. This can be easily done if the waves are of small amplitude. Using the linear dispersion relation for capillary waves,

$$q^* = \frac{\tau k^*}{\rho}\tanh(k^* h^*) \qquad (26)$$

where q^* is the flow speed in the region with waves, k^* is the wave number and h^* is the height of the water level above the horizontal bottom. Here * indicates that the parameters are dimensional.

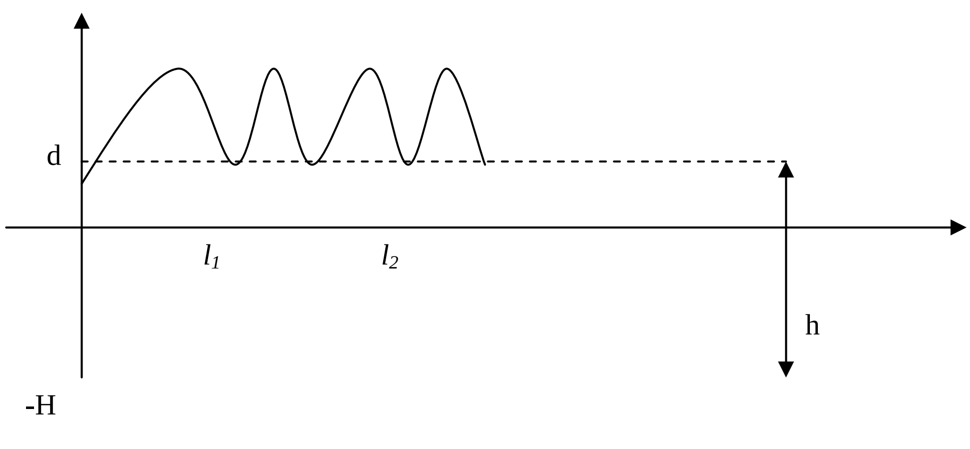

Figure 5: Free surface profile

Equating the mass fluxes to the left and right of the separation point where only waves of small amplitude are, consider,

$$\text{UH} = q^*(d^* + H)$$

or

$$q^* = \frac{UH}{d^* + H} \qquad (27)$$

combining equations (26) and (27), we obtain

$$\frac{U^2H^2}{(d^*+H)^2} = \frac{2\pi\tau}{\rho\lambda^*}\tanh(\frac{2\pi(d^*+H)}{\lambda^*}) \qquad (28)$$

where $\lambda^* = \frac{2\pi}{k^*}$ is the wavelength.

Non-dimensionalizing using the typical length *H* and velocity U_∞ gives:

$$\lambda = \frac{\lambda^*}{H}, \quad d = \frac{d^*}{H} \qquad (29)$$

substituting (29) into (28) gives:

$$\frac{U_\infty^2H^2}{(dH+H)^2} = \frac{2\pi\tau}{\rho\lambda H}\tanh[\frac{2\pi(dH+H)}{\lambda H}] \qquad (30)$$

Multiplying (30) by $\frac{1}{U_\infty^2}$ yields

$$\frac{1}{(d+1)^2} = \frac{\tau}{\rho H U_\infty^2}\frac{2\pi}{\lambda}\tanh[\frac{2\pi(d+1)}{\lambda}] \qquad (31)$$

where $\frac{\tau}{\rho H U_\infty^2} = \delta$ and d and δ are fixed parameters which can be used in equation (30) to solve for wavelength, λ.

Rewriting equation (31) gives

$$F(\lambda) = \frac{2\pi\delta}{\lambda}\tanh[\frac{2\pi(d+1)}{\lambda}] - \frac{1}{(d+1)^2} = 0 \qquad (32)$$

We use the root bisection method by introducing two approximations λ_1 and λ_2 of the wavelength λ, one being small and the other large respectively. To determine that the solution lies within the interval [λ_1, λ_2], it is necessary to check that

$$F(\lambda_1)F(\lambda_2) < 0 \qquad (33)$$

If the solution λ lie in the interval [λ_1, λ_2] due to change in sigh, then the function is evaluated at the midpoint of the interval. Again the sign is checked and the midpoint is used to replace whichever limit has the same sign. The process is repeated until the root has a significant accuracy.

Having known the wavelength, the uniform mesh is created. By making the mesh to be dependent on λ means that it is possible to know how many mesh points are in one wavelength, Vanden Broeck (1984). The mesh is created in the *f*-plane along the region representing the free surface, $\psi = 0$ and $0 \le \phi < \infty$ by

$$\phi_I = (I-1)\frac{\lambda}{n}, \text{ I = 1-------------N} \tag{34}$$

The distance between consecutive mesh points is $\frac{\lambda}{n}$ where *n* is the number of mesh points in one wavelength. We introduce midpoints at which τ can be calculated to avoid singularities. Let

$$\phi_I^m = \frac{\phi_{I+1} + \phi_I}{2}, \quad \text{I = 1--------------(N-1)} \tag{35}$$

and using (24), we have

$$\Gamma_I^m = \Gamma(\phi_I^m) = -\int_0^\infty \frac{\theta^1(\phi)e^{-\pi\phi}}{e^{-\pi\phi} - e^{-\pi\phi_I^m}} d\phi \tag{36}$$

where the right hand side of equation (36) is a Cauchy principle value integral and can be approximated using the trapezoidal rule with a summation over ϕ_I. This gives:

$$\Gamma_I^m = -\sum_{j=1}^{N} \frac{\theta_j e^{-\pi\phi_j}}{e^{-\pi\phi_j} - e^{-\pi\phi_I^m}} \frac{\lambda w_j}{n}, \text{ I = 1---------------------(N-1)} \tag{37}$$

where $w_j = \frac{1}{2}$ for *j = 1, N* and $w_j = 1$ otherwise, $\theta_I = \theta'(\phi_I)$. Substituting (37) into (23) gives a system of N-1 nonlinear equations in N-1 unknowns, θ_I for I = 1-----------N,

We need two more equations to complete the problem. One equation will come from fixing the separation angle θ_I and the second equation comes from fixing the height of the mean of the wave *d*. Using equation (25) gives

$$d = \frac{1}{\lambda}\int_{\phi-n}^{\phi} y x_\phi d\phi \tag{38}$$

Now the system has (N+1) equations with (N+1) unknowns and hence can be solved using Newton's method. Newton's method for solving nonlinear equations is one of the most well known procedures in numerical methods, Agoshkov and Ambrosi, (1993). This method uses not only the function values but also the first derivative to help find the next approximation to the root. It can be extended to solve systems of nonlinear equations as follows:

Let $$h = \frac{1}{2}e^{2\Gamma^1} - \delta e^{\Gamma^1}\frac{\partial\theta^1}{\partial\phi} - C^* \tag{39}$$

The system can be rewritten as

$$\begin{aligned} &h_1(\theta_1 ----------\theta_N, C^*, \alpha_B) = 0 \\ &h_2(\theta_1 ----------\theta_N, C^*, \alpha_B) = 0 \\ &h_{N+1}(\theta_1 ----------\theta_N, C^*, \alpha_B) = 0 \end{aligned} \tag{40}$$

It is necessary to find $\tilde{\theta}_1 ----------\tilde{\theta}_N, \tilde{C}^*, \tilde{\alpha}_B$ such that the system in equation (40) is satisfied. Let $C^* = \theta_N$ and $\alpha_B = \theta_{N+1}$ where $h_1 --------h_{N-1}$ are given by equation (39), where

$$\Gamma_I^m = \frac{1}{2}\ln\left|\frac{e^{-\pi\phi_B} - e^{-\pi\phi_I^m}}{1 - e^{-\pi\phi_I^m}}\right| - \sum_{j=1}^{N} \frac{\theta_j e^{-\pi\phi_j}\Delta w_j}{e^{-\pi\phi_j} - e^{-\pi\phi_I^m}}, \text{ I = 1---------N} \tag{41}$$

and

$$\frac{\partial\theta}{\partial\phi} = \frac{\theta_{I+1} - \theta_I}{\Delta}, \text{ I = 1---------------------(N-1)} \tag{42}$$

$\Delta > 0$ is the uniform increment in ϕ between consecutive mesh points, then

$$h_1 = \frac{1}{2}e^{2\Gamma_I^m} - \delta e^{\Gamma_I^m}\left(\frac{\theta_2 - \theta_1}{\Delta}\right) - C^* \tag{43}$$

$h_{N-1} = \alpha_B - \overline{\alpha}_B$ and $h_N = \theta_k - \overline{\theta}_k$ where $\overline{\alpha}_B$ and $\overline{\theta}_k$ are prescribed and Γ is given by equation (41). Let $\theta_1^{(n)} ------------\theta_{N+1}^{(n)}$ be an approximate solution. Each of the functions $h_1 ----h_{N+1}$ can be expanded about the approximate solution to give:

$$h_I(\tilde{\theta}_1 ---\tilde{\theta}_{N+1}) = 0 = h_I(\theta_1^{(n)} ---\theta_{N+1}^{(n)} + \sum \frac{\partial h_I}{\partial\theta_j}(\tilde{\theta}_j - \theta_j^{(n)}) + o(\tilde{\theta}_j - \theta_j^{(n)})^2 \tag{44}$$

Neglecting the higher order terms of equations (44) and let $\tilde{\theta}_j = \theta_j^{(n+1)}$, then equation (43) becomes:

$$-h_I(\theta_1^{(n)} ------\theta_{N+1}^{(n)}) = \sum_{j=1}^{N+1} \frac{\partial h_I}{\partial\theta_j}(\tilde{\theta}_j^{(n+1)} - \theta_j^{(n)}) \tag{45}$$

By introducing the Jacobian matrix J,

$$J = \begin{bmatrix} \frac{\partial h_1}{\partial \theta_1^{(n)}} - - - - - - - - - - - - \frac{\partial h_{N+1}}{\partial \theta_{N+1}^{(n)}} \\ \\ \frac{\partial h_1}{\partial \theta_{N+1}^{(n)}} - - - - - - - - - - - \frac{\partial h_{N+1}}{\partial \theta_{N+1}^{(n)}} \end{bmatrix} \quad (46)$$

Equation (45) can be written in matrix form as

$$-h^1 = J\theta^1$$

$$\theta^1 = -J^{-1}h^1 \quad (47)$$

with $\theta^1 = \theta^{1n} - (\theta^1)^{n-1}$ which gives rise to the Newton iteration process, Hoffman (1992):

$$(\theta^1)^{(n)} = (\theta^1)^{n-1} - J^{-1}h^1(\theta_1)^{n-1} - - - \theta_N^{(n)}, C^{*(n)}, \alpha_B^{(n)}, \text{ n =1,2,3} \quad (48)$$

The Jacobian equation (46) is evaluated by finite differences and then inverted by Gausssian elimination. Once the system (48) is solved, the unknowns θ_I must be used to transform from the *f*-plane back to the Cartesian plane. Using the real and imaginary parts of (13) gives the following identities:

$$\frac{\partial x}{\partial \phi} = e^{-\Gamma} \cos\theta \quad (49)$$

$$\frac{\partial y}{\partial \phi} = e^{-\Gamma} \sin\theta \quad (50)$$

Integrating (49) and (50) gives the Cartesian coordinates necessary to view the free surface profile.

For waves of large amplitude, the numerical scheme changes because we cannot fix the ordinate *d*.

Waveless Solution

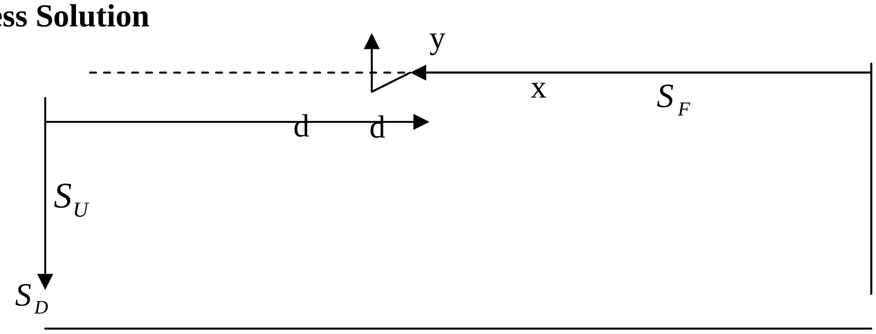

Figure 6: The components of the surface S

Here we need to show that for a non-zero separation angle, a waveless solution does not exist, Agoshkov and Ambrosi, (1993). The conservation of momentum principle will be used again to show this and the analytical work done will use Figure 6.

Let $y = d$ be the height of the mean elevation of the wave, let $y = 0$ be the level of the object and $y = -H$ be the bottom. The Bernoulli equation for the flow region is

$$\frac{1}{2}q^{*2} + \frac{p^*}{\rho} = C^* \tag{51}$$

$$q^{*2} = u_x^{*2} + u_y^{*2}$$

Using the conservation of momentum and neglecting body forces, we have

$$\int_s - p\mathbf{n}d\sigma = \int_s \rho\mathbf{u}(\mathbf{u.n})d\sigma \tag{52}$$

where S is any simple closed surface, ***n*** is the outward normal to that surface, σ is the arc length and u^* is the velocity vector. Considering only the *x*-component of equation (52) implies that

$$\int_S - pn_2 d\sigma = \int_S \rho u^* (\text{u.n})d\sigma \tag{53}$$

Multiplying equation (53) by $\frac{1}{\rho}^*$ and rearranging, it becomes

$$\int_S [u^*(\text{u.n}) + \frac{p^*}{\rho}n_x]d\sigma = 0 \tag{54}$$

We split the surface S into five parts, $S = S_F + S_D + S_B + S_U + S_p$, figure 5. Then

$$\int_{S_B} u^*(u.n) + \frac{p*}{\rho}n_x]d\sigma = \int_{Sp}[u^*(u.n) + \frac{p^*}{\rho}n_x]d\sigma = 0 \tag{55}$$

since **u.n**=0 and $n_x = 0$ along these surfaces. We now evaluate the remaining three integrals.

Along the surface S_U, $n_x = -1$ and $u^* = U$. Using equation (54) gives

$$\int_{S_U}[u^*(\text{u.n}) + \frac{p^*}{\rho}n_x]d\sigma = -\int_{-H}^{0}(U^2 + \frac{p^*}{\rho})dy \tag{56}$$

Along the surface $s_D, n_x = 1$ and $u^* = \overline{U}$. Using (54) gives

$$\int_{s_D}[u^*(\mathrm{u.n})+\frac{p^*}{\rho}n_x]d\sigma=\int_{-H}^{d}[\overline{U}^2+\frac{p^*}{\rho}]dy \tag{57}$$

where $\overline{U}$ is the downstream velocity as $x\to\infty$.

Along the surface s_F, $\mathbf{u.n}=0$. Using equation (54) gives

$$\int_{s_F}[u^*(\mathrm{u.n})+\frac{p^*}{\rho}n_x]d\sigma=\int_{\infty}^{0}\frac{p^*}{\rho}n_x d\sigma=\frac{1}{\rho}\int_{0}^{\theta_1}\frac{\tau}{R}(-\sin\theta)Rd\theta=\frac{\tau}{\rho}(\cos\theta_1-1) \tag{58}$$

Using (56), (57) and (58), (54) can be rewritten in terms of s_U, s_D and s_F to give

$$\int_{-H}^{d}(\overline{U}^2+\frac{p^*}{\rho})dy-\int_{-H}^{0}(U^2+\frac{p^*}{\rho})dy-\frac{\tau}{\rho}(1-\cos\theta_1)=0 \tag{59}$$

The second integral in equation (59) can be rewritten as

$$\int_{-H}^{0}(U^2+\frac{p^*}{\rho})dy=\int_{-H}^{0}(\frac{1}{2}U^2+(\frac{1}{2}U^2+\frac{p^*}{\rho})dy \tag{60}$$

This is done so that equation (51) can be used to evaluate the integral giving

$$\int_{-H}^{0}\frac{1}{2}U^2+(\frac{1}{\rho 2}U^2+\frac{p^*}{\rho})dy=\int_{-H}^{0}(\frac{1}{2}U^2+C^*)dy=H(\frac{1}{2}U^2+C^*) \tag{61}$$

The first integral can be evaluated in the same way to give:

$$\int_{-H}^{d}(\overline{U}^2+\frac{p^*}{\rho})dy=(d+H)(\frac{1}{2}\overline{U}^2+C^*) \tag{62}$$

Substituting equations (61) and (62) into equation (59) simplifies it to the second integral in equation (59) can be rewritten as

$$\int_{-H}^{0}(U^2+\frac{p^*}{\rho})dy=\int_{-H}^{0}(\frac{1}{2}U^2+(\frac{1}{2}U^2+\frac{p^*}{\rho})dy$$

This is done so that equation (51) can be used to evaluate the integral giving

$$\int_{-H}^{0}\frac{1}{2}U^2+(\frac{1}{\rho 2}U^2+\frac{p^*}{\rho})dy=\int_{-H}^{0}(\frac{1}{2}U^2+C^*)dy=H(\frac{1}{2}U^2+C^*) \tag{63}$$

The first integral can be evaluated in the same way to give:

$$\int_{-H}^{d}(\overline{U}^2+\frac{p^*}{\rho})dy=(d+H)(\frac{1}{2}\overline{U}^2+C^*) \tag{64}$$

Substituting equations (61) and (62) into equation (59) simplifies it to

$$(d+H)(\frac{1}{2}\overline{U}^2+C^*)-H(\frac{1}{2}U^2+C^*)-\frac{\tau}{\rho}(1-\cos\theta_1)=0 \tag{65}$$

multiplying equation (65) by $\dfrac{2}{HU^2}$ gives

$$\frac{2}{U^2}\left(\frac{H+d}{H}\right)\left(\frac{1}{2}\overline{U}^2+C^*\right)-H\left(\frac{1}{2}U^2+C^*\right)-\frac{2\tau}{\rho HU^2}(1-\cos\theta_1)=0 \qquad (66)$$

As x $\to\infty$, equation (51) implies that $C^*=\dfrac{\overline{U}^2}{2}$. Equating the mass fluxes for such a solution gives

$$UH=\overline{U}(H+d)$$

or

$$\frac{U}{\overline{U}}=\frac{H+d}{H}. \qquad (67)$$

Substituting into equation (64) yields

$$\frac{2U}{\overline{U}}\frac{\overline{U}^2}{U^2}-\frac{2}{U^2}\left(\frac{1}{2}U^2+\frac{1}{2}\overline{U}^2\right)-2\delta(1-\cos\theta_1)=0 \qquad (68)$$

Rearranging equation (66) gives

$$-2\delta(1-\cos\theta_1)=\frac{\overline{U}^2}{U^2}-\frac{2\overline{U}}{U}+1 \qquad (69)$$

and

$$\cos\theta_1=1+\frac{1}{2\delta}-\left(\frac{\overline{U}}{U}-1\right)^2 \qquad (70)$$

Equation (69) is satisfied if $\theta_1=0$ and $U=\overline{U}$ and using equation (66) then *d =0*. This is known as the uniform stream. If $U\neq\overline{U}$, then $\cos\theta_1>1$ which is a contradiction and so such a waveless solution does not exist which is similar to Whitham (1974)

The Numerical Results and Application

This confirm that there is a three parameter family of solutions and the free surface profiles have a discontinuity at the separation point *B* ($\theta_1\neq 0$) and a train of waves in the far field. For large ϕ it was found there was no sensitivity to the choice of end point in truncating the summation in equation (37). This was checked by truncating further into the far field until there was no graphical difference in the solutions obtained.

Osborne and Stump (2000) calculated free surface profiles of jets leaving circular and rectangular channels. They included effects of gravity only. A comparison can be made to the rectangular channel case by calculating free surface profiles using this numerical scheme under particular conditions. First the separation angle θ_1 must be equal to zero so that there is a continuity in slope against *F* where *F* was defined by writing the equation of the free surface as

$$y = \varepsilon F_1(x) + o(\varepsilon^2) \tag{71}$$

where $\varepsilon = d/(d+h)$ and solutions are found by assuming $\varepsilon << 1$, linearizing and solving the resultant linear equations by the Weiener Hopf technique, Hoffman (1992). Since $\mathrm{d} \to 0$ as $\varepsilon \to 0$, ε can be defined as $\varepsilon = \dfrac{d}{H}$. They described their solutions in terms of a parameter *F* defined by

$$F = \frac{\tau}{\rho c^2 (H+d)} \to 0 \tag{72}$$

For $\varepsilon << 1$, d<<H, then $F = \delta$. To reproduce the two profiles of Dias and Vanden Broeck (1989), δ is chosen as δ =1 and δ =0 and solutions are computed as $\mathrm{d} \to 0$. The resultant profiles for $F_1(x)$ are shown in figure 7, in which the relationship between λ and F is displayed. These curves also agree with the results of Osborne and Stump (2000). In figures 8 and 9, waveless free surface profile when the surface tension parameter changes between 100 to 1.

Then the scheme is used to generate the three parameter family of solutions that have halves on the free surface and a discontinuity in slope at the separation point *B*. Profiles are shown in figures *10*, *11* and *12*. In figure *10*, the waves are close to linear sine waves. In figure *12*, the profile is a nonlinear wave with broad crests and sharper. In the numerical scheme, it is possible to fix the ordinate *d* when the waves are of small amplitude. It is necessary to find an alternative parameter to fix when the amplitude of the wave becomes too large.

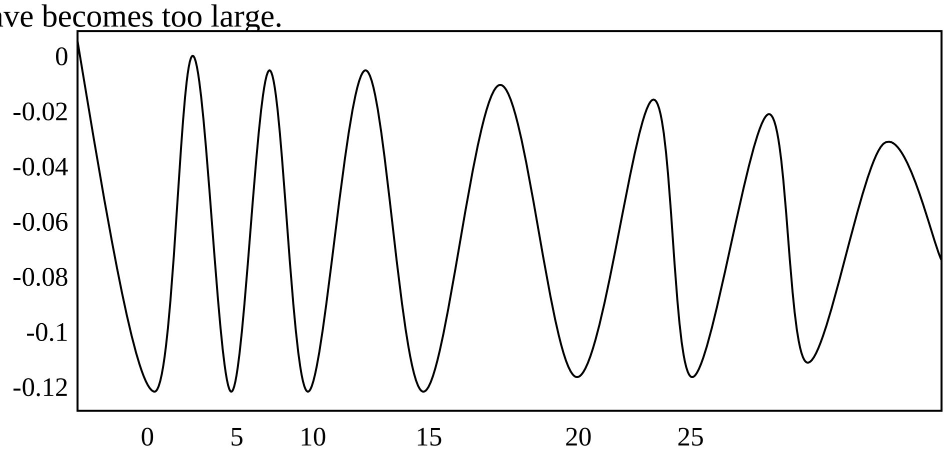

Figure 7: Analysis of accuracy of numerical method on a free surface profile

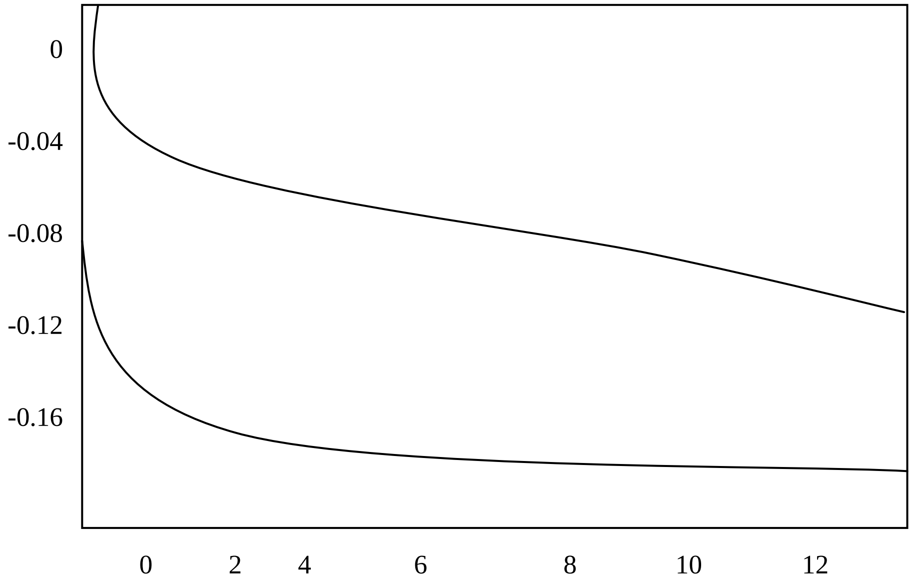

Figure 8: Waveless free surface profiles with $\alpha_B = 2.0$

From top to bottom τ =100, 10, 1

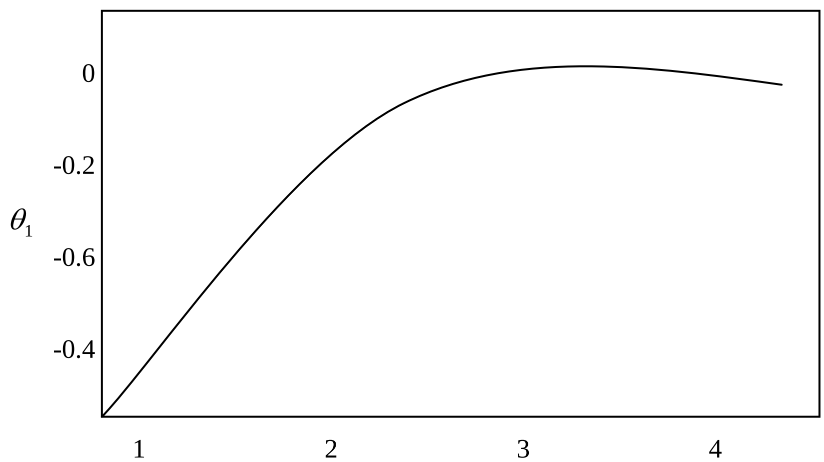

Figure 9: The change in θ_1 (radians) as the surface tension parameter τ changes

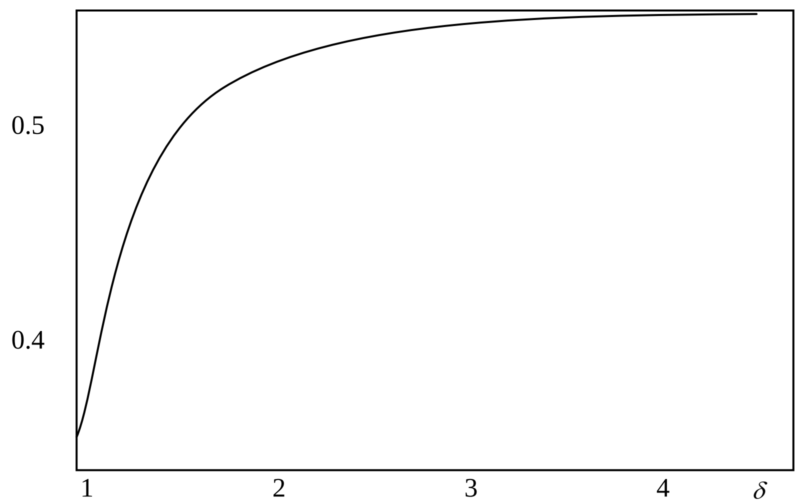

Figure 10: The change in contraction ratio as the surface tension parameter τ changes

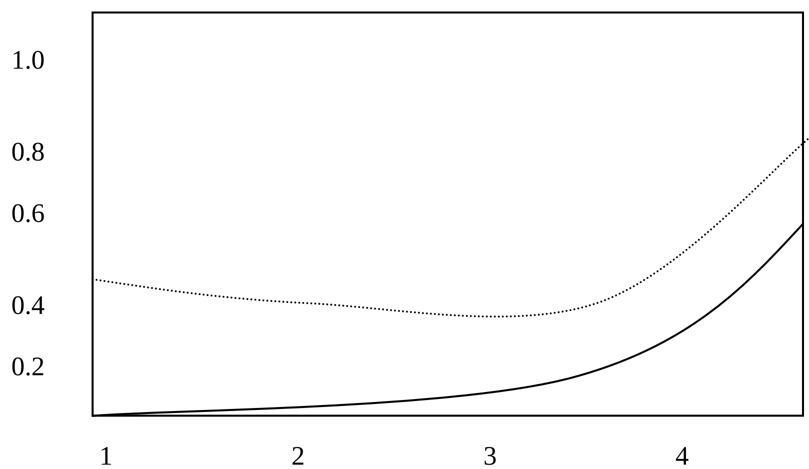

Figure 11: Comparison of the numerical results with Gurevich's results where .... Gurevich results,___Numerical results

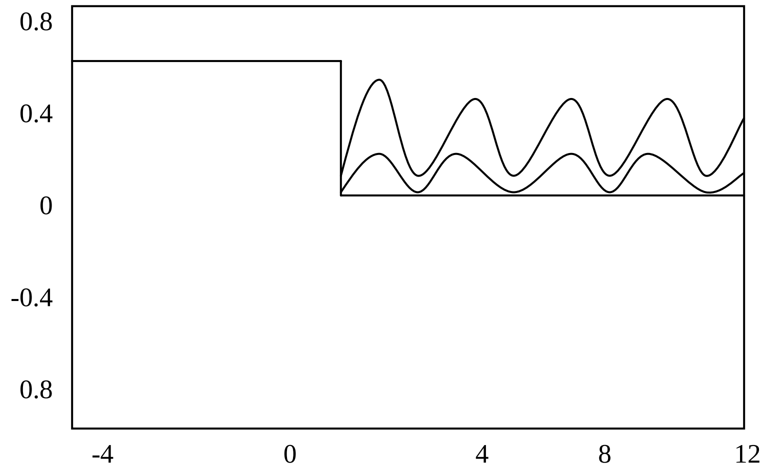

Figure 12: Surface profiles for nonlinear waves

Conclusions

We have considered the Boundary Integral Method to compute solutions for a free surface flow past a flat plate in a channel. The effects of surface tension was included in the boundary condition. It was discovered that there is a parameter family of solutions which include as a particular case the results computed by Osborne and Stump (2000). Also we have shown analytically that there are no nontrivial solutions. We have shown that there is a three parameter family of solutions when the fluid is of finite depth. These solutions are characterised by a train of waves in the downstream region and by a discontinuity in slope at the separation point. The family includes a two parameter sub- family for which the free surface leaves the plate tangentially. It is shown that this sub-family reduces to the linear solutions of Forbes and Schwarz (1982) when the depth of submergence of the object is small. Also, the three parameter family reduces to the one parameter family of Anderson and Vanden-Broeck (1996) as the depth of the water tends to infinity.

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